

## Renormalized field theory and particle density profile in driven diffusive systems with open boundaries

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We investigate the density profile in a driven diffusive system caused by a plane particle source perpendicular to the driving force. Focusing on the case of critical bulk density  $\bar{c}$  we use a field theoretic renormalization group approach to calculate the density  $c(z)$  as a function of the distance from the particle source at first order in  $\epsilon=2-d$  ( $d$  denotes the spatial dimension). For  $d=1$  we find reasonable agreement with the exact solution recently obtained for the asymmetric exclusion model. Logarithmic corrections to the mean field profile are computed for  $d=2$  with the result  $c(z) - \bar{c} \sim z^{-1} [\ln(z)]^{2/3}$  for  $z \rightarrow \infty$ . [S1063-651X(96)03105-4]

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### I. INTRODUCTION

It is well known that in thermodynamic systems with long-range correlations boundaries have a considerable influence on physical quantities even at macroscopic distances from the surface. During recent decades this effect has been investigated extensively in the context of surface critical phenomena (a review is given in Ref. [1]). It has been shown that the critical behavior near a boundary is governed by universal scaling laws with new critical exponents that cannot be expressed in terms of bulk exponents. The renormalization group has proved to be a useful method for the classification of both static [1–5] and dynamic [6–8] surface universality classes.

While in equilibrium systems long-range correlations occur if the thermodynamic parameters approach a critical point, they seem to be the rule in nonequilibrium systems with conserved dynamics [9,10]. In this paper we study the diffusion of particles subject to a driving force in a system with open boundaries. Here the particle conservation in conjunction with the deviation from detailed balance leads to long-range correlations and anomalous long-time behavior [11] even at temperatures  $T \gg T_c$  above the critical point (for a review see [10]). Especially interesting from a physical viewpoint are surfaces that act as particle reservoirs and thus break the conservation law. In the case of boundaries perpendicular to the driving force that we consider in the present paper a particle reservoir is necessary to maintain a nonvanishing steady current.

A simple microscopic realization of a driven diffusive system (DDS) is a lattice gas [12] with hard core repulsion and nearest neighbor hopping. In a homogeneous state of density (particles per lattice site)  $c$  the external field (favoring jumps in the positive  $z$  direction) produces a steady particle current  $j(c)$ . Due to the excluded volume constraint the current vanishes if every site is occupied by a particle, i.e.,  $j(1)=0$ , while  $j(c) \sim c$  for  $c \ll 1$ . In a work by Krug [13] on boundary-induced phase transitions the general form of the density profile has been discussed. It has been shown that the bulk density  $\bar{c}$  satisfies the following maximum current principle: If the boundary at  $z=0$  is in contact with a particle reservoir of density  $c(0)$  and every particle that reaches the

boundary at  $z=L$  leaves the system, i.e.,  $c(L)=0$ , then the current is maximized in the sense that

$$j(\bar{c}) = \max\{j(c) | 0 \leq c \leq c(0)\} \quad (1)$$

for  $L \rightarrow \infty$ . As a direct consequence of this principle the bulk density is at the maximum point  $c^*$  of the function  $j(c)$  and the profile decays only algebraically from boundary to bulk value if  $c(0) \geq c^* \geq c(L)$ .

Recently the exact density profile in one dimension has been calculated for arbitrary boundary conditions  $0 \leq c(0), c(L) \leq 1$  [14,15]. These works confirm and generalize a large part of the results obtained in Ref. [13].

Until now no exact solutions have been found for more complicated problems such as driven diffusion in higher dimensional systems, at a critical point [16], or in a medium with quenched disorder [17]. In these cases the field theoretic approach is useful in obtaining systematic approximations for density profiles or correlation and response functions. In the present paper we use the exact solution of the one-dimensional asymmetric exclusion model found in [14,15] to test the accuracy of the renormalization-group improved perturbation theory. We also extend the analysis to two-dimensional DDS.

In the next section we present the semi-infinite extension of the continuum model for DDS introduced in Ref. [11]. In Sec. III the boundary conditions are discussed and the density profile is calculated in a mean field approximation. Having introduced the renormalization factors that are required to obtain a well-defined renormalized field theory in Sec. IV we calculate the Gaussian fluctuations around the mean field profile. We use the renormalization group to compute the universal scaling function for the profile at first order in  $\epsilon=2-d$  and compare our result with the exact one-dimensional profile. For  $d=2$  we obtain the logarithmic corrections to the mean field solution. Section V contains a discussion of our findings and an outlook. In Appendix A some technical details of the calculation of the surface divergencies at one-loop order are given. In order to compare our perturbative result with the exact solution we have to take the continuum limit of the exact profile. This is done in Appendix B.

## II. THE MODEL

Some time ago, van Beijeren, Kutner, and Spohn [18] introduced a continuum model for the diffusion of particles subject to a driving force. The anomalous long time behavior of this DDS was studied by Schmittmann and one of us [11] by renormalization group methods. The equation of motion for the particle density  $c(\mathbf{r}, t)$  in this model is given by the continuity equation

$$\frac{\partial}{\partial t} c(\mathbf{r}, t) + \nabla \mathbf{j}(\mathbf{r}, t) = 0. \quad (2)$$

Here the current  $\mathbf{j}(\mathbf{r}, t)$  consists of a diffusive part, a contribution caused by the driving force  $\mathbf{E}$ , and a part  $\mathbf{j}_R(\mathbf{r}, t)$ , which is assumed to summarize the effects of the fast microscopic degrees of freedom:

$$\mathbf{j}(\mathbf{r}, t) = -\mathbf{D}\nabla c(\mathbf{r}, t) + \kappa(c(\mathbf{r}, t))\mathbf{E} + \mathbf{j}_R(\mathbf{r}, t). \quad (3)$$

This form of the current can—in principle—be derived from the microscopic dynamics by a suitable coarse graining. Since the external field  $\mathbf{E}$  introduces an anisotropy into the system the coarse graining in general gives rise to an anisotropic matrix of diffusion constants  $\mathbf{D}$ .

Expanding the conductivity  $\kappa(c)$  in the deviation  $s = c - \bar{c}$  of the density from its uniform average (bulk) density  $\bar{c}$ , we have

$$\kappa(c) = \kappa(\bar{c}) + \kappa'(\bar{c})s + \frac{1}{2}\kappa''(\bar{c})s^2 + \dots \quad (4)$$

Neglecting higher order terms in this expansion one obtains the Langevin equation

$$\begin{aligned} \partial_t s(\mathbf{r}, t) = & \lambda(\Delta_{\perp} + \rho\partial_{\parallel}^2)s(\mathbf{r}, t) + \lambda\partial_{\parallel}\left(f s(\mathbf{r}, t) + \frac{1}{2}g s(\mathbf{r}, t)^2\right) \\ & + \zeta(\mathbf{r}, t), \end{aligned} \quad (5)$$

where  $f \propto -E\kappa'(\bar{c})$  and  $g \propto -E\kappa''(\bar{c})$ , and the indices  $\parallel$  and  $\perp$  distinguish spatial directions parallel (“longitudinal”) and perpendicular (“transverse”) to the driving force. The Langevin force  $\zeta = -\nabla \mathbf{j}_R$  is assumed to be Gaussian with zero mean and the correlations (after a suitable rescaling of  $s$ )

$$\langle \zeta(\mathbf{r}, t)\zeta(\mathbf{r}', t') \rangle = -2\lambda(\Delta_{\perp} + \sigma\partial_{\parallel}^2)\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'). \quad (6)$$

It was shown in [11] that the model defined by Eqs. (5) and (6) is complete in the renormalization group sense; i.e., further contributions to (5) and (6) as well as non-Gaussian and non-Markovian parts of the Langevin forces are irrelevant for the long-time and large-distance properties of the system as long as the diffusion constants are positive.

For an infinite system the term proportional to the coupling constant  $f$  in the Langevin equation (5) can be eliminated by a suitable Galilean transformation [11]. Such a Galilean transformation can of course not be applied to a system with time independent surfaces perpendicular to the driving force. However, Krug has shown [13] that in a system with open boundaries in which the particles are driven from a reservoir of density  $c_1$  to a second reservoir of density  $c_2$

with  $c_1 > c^* > c_2$  the particle density far in the bulk takes the value  $c^*$ , which maximizes the function  $\kappa(c)$ . In this case we have  $f \propto -\kappa'(c^*) = 0$  and  $g \propto -\kappa''(c^*) > 0$ , and the density decreases only algebraically from the boundary value to the bulk value. Since we are interested in the behavior of the system near one of the boundaries we may effectively consider a semi-infinite system with bulk density  $\bar{c} = c^*$ , i.e.,  $f = 0$ .

Appropriate boundary conditions for diffusive semi-infinite systems have been derived in [7,8]. This approach crucially rests on the assumption of detailed balance. In Ref. [11] it was shown that the dynamics defined by the bulk stochastic equations of motion (5,6) with  $f = 0$  satisfies this assumption if  $w = \sigma/\rho = 1$ . Under renormalization group transformations the coupling constant  $w$  flows to the detailed balance value  $w_* = 1$  if  $d \leq 2$ . For  $d > 2$  the system displays mean-field-like behavior. It should be remarked that for  $w \neq 1$  the correlations are long ranged [19]. Thus for  $d \leq 2$  the variable  $(w - 1)$  plays the role of a dangerously irrelevant coupling and its effects have to be studied.

To set up a renormalized field theory it is convenient to recast the bulk model in terms of the dynamic functional [20–25,11]

$$\begin{aligned} \mathcal{J}_b[\tilde{s}, s] = & \int dt \int_V d^d r \left\{ \tilde{s} \partial_t s + \lambda \left[ (\nabla_{\perp} \tilde{s})(\nabla_{\perp} s) + \rho(\partial_{\parallel} \tilde{s})(\partial_{\parallel} s) \right. \right. \\ & \left. \left. + f(\partial_{\parallel} \tilde{s})s + \frac{1}{2}g(\partial_{\parallel} \tilde{s})s^2 - (\nabla_{\perp} \tilde{s})^2 - \sigma(\partial_{\parallel} \tilde{s})^2 \right] \right\}, \end{aligned} \quad (7)$$

where  $\tilde{s}$  is a Martin-Siggia-Rose response field [26]. The range of integration with respect to  $\mathbf{r}$  is the  $d$ -dimensional half space  $V = \{\mathbf{r} = (\mathbf{r}_{\perp}, z) | \mathbf{r}_{\perp} \in \mathbf{R}^{d-1}, z \geq 0\}$ . Correlation and response functions in the bulk can now be expressed as functional averages with weight  $\exp(-\mathcal{J}_b)$ . For  $w = \sigma/\rho = 1$  the functional  $\mathcal{J}_b$  can be written in the detailed balance form [24,25]

$$\mathcal{J}_b[\tilde{s}, s] = \int dt \int_V d^d r \left\{ \tilde{s} \left[ \partial_t s + \mathbf{R}_b \left( \frac{\delta \mathcal{H}}{\delta s} - \tilde{s} \right) \right] \right\}, \quad (8)$$

with the reaction kernel

$$\mathbf{R}_b = \lambda \left[ (\tilde{\nabla}_{\perp} \nabla_{\perp} + \rho \tilde{\partial}_{\parallel} \partial_{\parallel}) + \frac{g}{3}(\tilde{\partial}_{\parallel} s - s \partial_{\parallel}) \right]. \quad (9)$$

Here  $\tilde{\nabla}$  and  $\tilde{\partial}$  act to the left, while  $\nabla$  and  $\partial$  act as usual to the right. The Hamiltonian

$$\mathcal{H}[s] = \int_V d^d r \frac{1}{2} s(\mathbf{r})^2 \quad (10)$$

defines the purely Gaussian stationary state distribution. Obviously,  $\mathcal{J}_b$  now obeys the detailed balance symmetry [24,25]

$$s(t) \rightarrow -s(-t),$$

$$\tilde{s}(t) \rightarrow \tilde{s}(-t) - \left. \frac{\delta \mathcal{H}}{\delta s} \right|_{-t} = \tilde{s}(-t) - s(-t). \quad (11)$$

The last equation implies in particular that

$$\langle s(\mathbf{r}, t) \tilde{s}(\mathbf{r}', t') \rangle = \Theta(t - t') \langle s(\mathbf{r}, t) s(\mathbf{r}', t') \rangle. \quad (12)$$

We now turn to the surface contributions  $\mathcal{F}_1$  in the full dynamic functional  $\mathcal{F} = \mathcal{F}_b + \mathcal{F}_1$  arising from the boundary layer at  $z=0$ . Locality is assumed for the bulk and the surface. Therefore  $\mathcal{F}_1$  should be written as a  $(d-1)$ -dimensional integral over surface fields alone. It is

$$\begin{aligned} \mathcal{F}[\tilde{s}, s] = & \int dt \int d^d r \left\{ \tilde{s} \left[ \partial_t s + \mathbf{R} \left( \frac{\delta \mathcal{H}}{\delta s} - \tilde{s} \right) \right] \right\} = \int dt \int_V d^d r \{ \tilde{s} \partial_t s + \lambda [(\nabla_\perp \tilde{s})(\nabla_\perp s) + \rho(\partial_\parallel \tilde{s})(\partial_\parallel s)] + \frac{1}{2} \lambda g(\partial_\parallel \tilde{s}) s^2 \\ & - \lambda [(\nabla_\perp \tilde{s})^2 + \rho(\partial_\parallel \tilde{s})^2] \} + \int dt \int_{\partial V} d^{d-1} r_\perp \lambda [ \tilde{c} \tilde{s}(s - \tilde{s}) + b \tilde{s}(s - \tilde{s})(s - 2\tilde{s}) - \frac{1}{6} g \tilde{s} s^2 ]. \end{aligned} \quad (14)$$

We now consider the modifications that we expect if detailed balance does not hold. First of all, the bulk functional is given by (7) with a noise constant  $\sigma$  independent of the diffusion constant  $\rho$ . In addition, the different surface fields that show up in (14) are now independent, and a boundary source has to be added to the equation of motion (5) leading to a term linear in  $\tilde{s}_1$ . We thus get for the surface dynamic functional

$$\begin{aligned} \mathcal{F}_1[\tilde{s}, s] = & \int dt \int_{\partial V} d^{d-1} r_\perp \lambda \left( c \tilde{s} s - \tilde{c} \tilde{s}^2 - \tilde{h}_1 \tilde{s} + \frac{1}{2} g_a \tilde{s} s^2 \right. \\ & \left. - \frac{1}{2} g_b \tilde{s}^2 s + \frac{1}{6} g_c \tilde{s}^3 \right). \end{aligned} \quad (15)$$

In Eqs. (14) and (15) we have omitted the surface operators  $s \partial_n \tilde{s}$ ,  $\tilde{s} \partial_n s$ ,  $\tilde{s} \partial_n \tilde{s}$ ,  $\partial_n \tilde{s}$ , and  $\partial_n^2 \tilde{s}$  (where  $\partial_n = \partial_\parallel$  means the normal derivative) since they can be expressed in terms of the composite fields retained in  $\mathcal{F}_1$  (see also below).

We remark that we have always used the prepoint time discretization in the construction of the dynamic functional [24,25]. This corresponds to the definition  $\Theta(t=0)=0$  and thus allows us to omit all (measure) terms  $\propto \Theta(0)$  in  $\mathcal{F}$ .

### III. EQUATIONS OF MOTION AND MEAN FIELD PROFILE

Introducing the functional

$$\begin{aligned} Z[\tilde{J}, J; \tilde{J}_1, J_1] = & \int \mathcal{A}[\tilde{s}, s] \exp[-\mathcal{F}_b[\tilde{s}, s] - \mathcal{F}_1[\tilde{s}, s, s_s] \\ & + (\tilde{J}, \tilde{s}) + (J, s) + (\tilde{J}_1, \tilde{s}_s) + (J_1, s_s)] \end{aligned} \quad (16)$$

we may write correlation and response functions as derivatives of  $Z$  with respect to the bulk sources  $\tilde{J}$ ,  $J$  and the

easily seen that for the Hamiltonian  $\mathcal{H}$  only irrelevant surface terms can be constructed. Thus  $\mathcal{H}$  has the form (10) also in the semi-infinite case. To retain the detailed balance form (9) of  $\mathcal{F}$  we can only modify the reaction kernel (9)  $\mathbf{R}_b \rightarrow \mathbf{R} = \mathbf{R}_b + \mathbf{R}_1$ . Such a modification has to describe the breakdown of particle conservation due to the reservoir [8]. The relevant contributions that respect the symmetry (11) are

$$\mathbf{R}_1 = \lambda [\tilde{c} + b(s - 2\tilde{s})] \delta(z) \quad (13)$$

up to redundant terms. Since the kernel  $\mathbf{R}$  has to be positive for reasons of stability it follows that  $\tilde{c} > 0$ . This completes our construction of the dynamic functional in the case of detailed balance. We eventually get

surface sources  $\tilde{J}_1$ ,  $J_1$ . In Eq. (16),  $\tilde{s}_s$  and  $s_s$  denote the surface fields and we have used the abbreviations

$$(\tilde{J}, \tilde{s}) = \int dt \int_V d^d r \tilde{J} \tilde{s} \quad (17)$$

and

$$(J_1, s_s) = \int dt \int_{\partial V} d^{d-1} r_\perp J_1 s_s. \quad (18)$$

To obtain the boundary conditions that are determined by the surface functional  $\mathcal{F}_1$  we exploit the invariance of  $Z$  with respect to a shift of the fields  $\tilde{s}$  and  $s$ . This invariance implies the equations of motion

$$\langle \delta \mathcal{F}_A \rangle_{\{\tilde{J}, J; \tilde{J}_1, J_1\}} = J_A \text{ for } A = \tilde{s}, s, \tilde{s}_s, s_s, \quad (19)$$

with the notation  $J_s = J$ ,  $J_{\tilde{s}} = \tilde{J}$ ,  $J_{s_s} = J_1$ , and  $J_{\tilde{s}_s} = \tilde{J}_1$ , respectively, and

$$\begin{aligned} \delta \mathcal{F}_{\tilde{s}} = & \partial_t \tilde{s} - \lambda [(\Delta_\perp + \rho \partial_\parallel^2) \tilde{s} + f \partial_\parallel s + \frac{1}{2} g \partial_\parallel s^2 \\ & - 2(\Delta_\perp + \sigma \partial_\parallel^2) \tilde{s}], \end{aligned} \quad (20a)$$

$$\delta \mathcal{F}_s = -\partial_t \tilde{s} - \lambda [(\Delta_\perp + \rho \partial_\parallel^2) \tilde{s} - f \partial_\parallel \tilde{s} - g(\partial_\parallel \tilde{s}) s], \quad (20b)$$

$$\begin{aligned} \delta \mathcal{F}_{\tilde{s}_s} = & \lambda [-\rho \partial_n s + 2\sigma \partial_n \tilde{s} + (c - f) s_s - 2\tilde{c} \tilde{s}_s + \frac{1}{2} (g_a - g) s_s^2 \\ & - g_b \tilde{s}_s s_s + \frac{1}{2} g_c \tilde{s}_s^2 - \tilde{h}_1], \end{aligned} \quad (20c)$$

$$\delta \mathcal{F}_{s_s} = \lambda (-\rho \partial_n \tilde{s} + c \tilde{s}_s + g_a \tilde{s}_s s_s - \frac{1}{2} g_b \tilde{s}_s^2). \quad (20d)$$

Equations (19), (20c), and (20d) show that  $\partial_n \tilde{s}$  and  $\partial_n s$  can be expressed in terms of the surface operators included in

$\mathcal{J}_b$ . The redundancy of  $\tilde{s}_s \partial_n s$ ,  $s_s \partial_n \tilde{s}$ , and  $\tilde{s}_s \partial_n \tilde{s}$  follows by differentiating  $\langle \partial_n \tilde{s} \rangle$  and  $\langle \partial_n s \rangle$  with respect to  $\tilde{J}_1$  and  $J_1$ , respectively.

Setting  $\tilde{J}=J=\tilde{J}_1=J_1=0$  in (19) we obtain from (20a) the equation

$$\partial_z [\rho \partial_z \Phi(z) + f \Phi(z) + \frac{1}{2} g \langle s(\mathbf{r}_\perp, z; t)^2 \rangle] = 0 \quad (21)$$

for the steady density profile  $\Phi(z) = \langle s(\mathbf{r}_\perp, z; t) \rangle$ . Using the boundary condition  $\Phi_{\text{bulk}} = 0$ , which follows from our definition of  $s$ , an integration with respect to  $z$  yields

$$\rho \Phi'(z) + f \Phi(z) + \frac{1}{2} g [C(z) - C_{\text{bulk}} + \Phi(z)^2] = 0, \quad (22)$$

where the function  $C(z) = \langle [s(\mathbf{r}_\perp, z; t) - \Phi(z)]^2 \rangle$  (with the bulk value  $C_{\text{bulk}}$ ) describes density fluctuations. The boundary condition at  $z=0$  follows from (20c) and (22) as

$$c \Phi_0 + \frac{1}{2} g_a [C(0) + \Phi_0^2] - \frac{1}{2} g C_{\text{bulk}} = \tilde{h}_1. \quad (23)$$

Note that equal time expectation values such as  $\langle s(t)^n \tilde{s}(t)^m \rangle$  with  $m > 0$  are zero as a consequence of our prepoint time discretization. The function  $C(z)$  contains an ultraviolet divergence proportional to  $\delta(\mathbf{0}) = \Lambda_\perp^{d-1} \Lambda_\parallel$  (where  $\Lambda_\perp$  and  $\Lambda_\parallel$  are cutoff wave numbers), which cancels in Eq. (22) for  $z > 0$ .

The mean field profile is the solution of Eq. (22) for vanishing fluctuations, i.e.,  $C(z) = 0$ . In the case  $f=0$ , which describes the maximum current phase, we obtain

$$\Phi_{\text{mf}}(z) = \Phi_0 \left( 1 + \frac{g}{2\rho} \Phi_0 z \right)^{-1}. \quad (24)$$

#### IV. RENORMALIZATION GROUP ANALYSIS

##### A. Surface divergencies

In the next subsection the density profile will be calculated by an expansion around the mean field profile (24). Since the perturbative calculation of Green functions leads to integrals that are ultraviolet divergent in the upper critical dimension  $d_c = 2$ , we have to employ a regularization method to obtain a well-defined perturbation series. We utilize the method of dimensional regularization that allows us to render all integrals finite up to poles in  $\epsilon = 2 - d$ . The  $\epsilon$  poles are then absorbed into renormalizations of coupling coefficients and fields (minimal renormalization). It was shown in Ref. [11] that for the bulk system a renormalization of the coupling coefficients  $\rho$  and  $\sigma$  is sufficient to cancel all uv divergencies.

In general the breaking of translational invariance by a boundary generates additional divergencies that require additional renormalizations of surface couplings and fields. (For a review see Ref. [1].) To determine the surface divergencies we first consider the flat profile  $\Phi(z) = 0$ , which is a steady state of the system provided that  $\Phi_0 = 0$  and  $C(z) = C_{\text{bulk}}$  for all  $z > 0$  [see Eq. (22)]. [At the stable fixed point with  $w = \sigma/\rho = 1$  the  $z$  independence of  $C(z)$  follows immediately from the the Hamiltonian (10).] It will be shown below that this condition is generally satisfied for  $d=1$  and in the high-temperature, strong-field limit in higher dimensions.

The Fourier transform  $\hat{G}_{\mathbf{q}_\perp, \omega}(z, z')$  of the free (Gaussian) propagator  $\langle s(\mathbf{r}_\perp, z; t) \tilde{s}(\mathbf{0}, z'; 0) \rangle$  is the solution of the differential equation

$$(i\omega + \lambda q_\perp^2 - \lambda \rho \partial_z^2) \hat{G}_{\mathbf{q}_\perp, \omega}(z, z') = \delta(z - z') \quad (25)$$

with the boundary condition

$$\rho \partial_z \hat{G}_{\mathbf{q}_\perp, \omega}(z, z')|_{z=0} = c \hat{G}_{\mathbf{q}_\perp, \omega}(z, 0). \quad (26)$$

This follows in a straightforward way from the terms bilinear in  $\tilde{s}$  and  $s$  of the functional  $\mathcal{J}_b$  (with  $f=0$ ) and Eqs. (19) and (20d). The solution is given by

$$\hat{G}_{\mathbf{q}_\perp, \omega}(z, z') = \frac{1}{2\lambda \sqrt{\rho} \kappa} \left[ e^{-\kappa|z-z'|/\sqrt{\rho}} + \frac{\kappa - c/\sqrt{\rho}}{\kappa + c/\sqrt{\rho}} e^{-\kappa(z+z')/\sqrt{\rho}} \right] \quad (27)$$

with

$$\kappa = \left( \frac{i\omega}{\lambda} + q_\perp^2 \right)^{1/2}. \quad (28)$$

The parameter  $c$  occurring in the surface functional  $\mathcal{J}_1$  and in the propagator describes (for  $c > 0$ ) the suppression of density fluctuations by the particle reservoir at the boundary  $z=0$ . Since its momentum dimension is  $c \sim \sqrt{\rho} \mu$  the asymptotic scaling behavior is governed by the fixed point  $c_* = \infty$ . This means that up to corrections to scaling the boundary condition (26) may be replaced by the Dirichlet boundary conditions  $\tilde{s}(z=0) = 0$  and  $s(z=0) = 0$ .

The Fourier transform of the Gaussian correlator  $\langle s(\mathbf{r}_\perp, z; t) s(\mathbf{0}, z'; 0) \rangle$  at the Dirichlet fixed point follows from (7) as

$$\hat{C}_{\mathbf{q}_\perp, \omega}(z, z') = 2\lambda \int_0^\infty dy \hat{G}_{\mathbf{q}_\perp, \omega}(z, y) \times (q_\perp^2 + \sigma \tilde{\partial}_y \partial_y) \hat{G}_{-\mathbf{q}_\perp, -\omega}(z', y). \quad (29)$$

Using the differential equation (25) for the propagator and the Dirichlet boundary conditions this may be written in the form

$$\hat{C}_{\mathbf{q}_\perp, \omega}(z, z') = w [\hat{G}_{\mathbf{q}_\perp, \omega}(z, z') + \hat{G}_{-\mathbf{q}_\perp, -\omega}(z', z)] + 2(1-w) \lambda q_\perp^2 \int_0^\infty dy \hat{G}_{\mathbf{q}_\perp, \omega}(z, y) \times \hat{G}_{-\mathbf{q}_\perp, -\omega}(z', y), \quad (30)$$

where  $w = \sigma/\rho$ . For the calculation of density fluctuations it is useful to transform the correlator into  $(\mathbf{q}_\perp, z, t)$  representation. The result reads

$$C_{\mathbf{q}_\perp}(z, z'; t) = w G_{\mathbf{q}_\perp}(z, z'; |t|) + (1-w)\lambda q_\perp^2 \times \int_{|t|}^{\infty} G_{\mathbf{q}_\perp}(z, z'; t') dt', \quad (31)$$

where

$$G_{\mathbf{q}_\perp}(z, z'; t) = \Theta(t) (4\pi\lambda\rho t)^{-1/2} e^{-\lambda q_\perp^2 t} \left[ \exp\left(-\frac{(z-z')^2}{4\lambda\rho t}\right) - \exp\left(-\frac{(z+z')^2}{4\lambda\rho t}\right) \right] \quad (32)$$

is the free propagator.

For  $w=1$  Eq. (31) reflects the detailed balance symmetry (12). Integrating the correlator (31) over  $q_\perp$  we obtain for  $t=0, z>0$

$$C(z) - C_{\text{bulk}} = -(1-w) \frac{(d-1)\Gamma(d/2)}{2\sqrt{\rho}(4\pi)^{d/2}} (z/\sqrt{\rho})^{-d}. \quad (33)$$

It is thus necessary for the flat profile to be stationary in dimensions  $d>1$  that the condition  $w=1$  is satisfied. Deviations from this stable fixed point generate the long-range correlations [9] in the bulk that are responsible for the  $z$  dependence of the function  $C(z)$ . In this sense ( $w-1$ ) is dangerously irrelevant. There is, however, a microscopic realization of DDS in which equal-time correlations are absent in spite of the violation of (microscopic) detailed balance [27]: In the high-temperature, strong-field limit the particles are noninteracting except for the excluded volume constraint and cannot jump against the direction of the driving field. The lack of correlations in this limit can be incorporated into the field-theoretic description by setting  $w=1$  from the outset. For  $d=1$  we can render  $w=1$  by a simple rescaling of coupling coefficients and fields.

We now proceed along the lines taken in the analysis of the surface critical behavior of equilibrium systems at the ordinary transition [1,2,28]. Since the correlator (31) and the propagator (32) vanish for  $z'=0$  or  $z=0$ , Green functions with insertions of the surface fields  $\tilde{s}_s$  and  $s_s$  tend to zero for  $c\rightarrow\infty$ . A convenient method to investigate the scaling behavior of quantities that vanish for  $c=\infty$  is the  $1/c$  expansion [2,28].

For large  $c$  the propagator of the surface field  $\tilde{s}_s$  behaves as

$$\hat{C}_{\mathbf{q}_\perp, \omega}(z, 0) = c^{-1} \rho \partial_z \hat{G}_{\mathbf{q}_\perp, \omega}^{(D)}(z, z')|_{z'=0} + O(c^{-2}) \quad (34)$$

for  $z>0$ . In order to derive an analogous relation for the correlator we have to take into account the surface coupling  $\tilde{c}$  in  $\mathcal{F}_1$ , which has the same  $\mu$  dimension as  $c$ . By differentiating the equation of motion (19) and (20c) with respect to  $J$  one obtains

$$\begin{aligned} \rho \partial_z \hat{C}_{\mathbf{q}_\perp, \omega}(z, z')|_{z'=0} &= 2\sigma \partial_z \hat{G}_{\mathbf{q}_\perp, \omega}(z, z')|_{z'=0} \\ &+ c \hat{C}_{\mathbf{q}_\perp, \omega}(z, 0) - 2\tilde{c} \hat{G}_{\mathbf{q}_\perp, \omega}(z, 0) \end{aligned} \quad (35)$$

and, for  $c\rightarrow\infty$ ,

$$\begin{aligned} \hat{C}_{\mathbf{q}_\perp, \omega}(z, 0) &= c^{-1} \rho [\partial_z \hat{C}_{\mathbf{q}_\perp, \omega}^{(D)}(z, z')|_{z'=0} \\ &- 2(w-w_s) \partial_z \hat{G}_{\mathbf{q}_\perp, \omega}^{(D)}(z, z')|_{z'=0}] + O(c^{-2}) \end{aligned} \quad (36)$$

(with  $w_s = \tilde{c}/c$ ). Therefore the leading order terms in an expansion in powers of  $c^{-1}$  can be studied in the framework of a field theory with Dirichlet boundary conditions by the replacements

$$\tilde{s}_s \rightarrow c^{-1} \rho \partial_n \tilde{s}, \quad (37a)$$

$$s_s \rightarrow c^{-1} \rho [\partial_n s - 2(w-w_s) \partial_n \tilde{s}] \quad (37b)$$

in expectation values. To investigate the scaling behavior of Green functions with insertions of  $\tilde{s}_s$  and  $s_s$  we have to compute the uv singularities generated by the surface fields on the right-hand side of (37). These singularities require renormalizations of the form

$$\rho \partial_n \tilde{s} = \tilde{Z}_1^{1/2} [\rho \partial_n \tilde{s}]_R, \quad (38a)$$

$$\rho (\partial_n s - 2w \partial_n \tilde{s}) = Z_1^{1/2} [\rho (\partial_n s - 2w \partial_n \tilde{s})]_R, \quad (38b)$$

$$\tilde{Z}_1^{1/2} w_s = Z_1^{1/2} w_{sR}. \quad (38c)$$

The renormalization factors  $\tilde{Z}_1$  and  $Z_1$  are determined by requiring that the averages

$$\langle [\rho \partial_n \tilde{s}]_R s(z>0) \rangle$$

and

$$\langle [\rho (\partial_n s - 2w \partial_n \tilde{s})]_R \tilde{s}(z>0) \rangle$$

be finite for  $\epsilon=0$ . Due to causality the term  $2\rho w \partial_n \tilde{s}$  gives no contribution to the second expectation value but in the Green function  $\langle \rho (\partial_n s - 2w \partial_n \tilde{s}) \cdot s(z>0) \rangle$  it is necessary to obtain a surface field that can be multiplicatively renormalized by  $Z_1$ .

The renormalizations that are necessary to cancel all uv divergencies in the translationally invariant bulk theory are given by [11]

$$\rho \rightarrow \hat{\rho} = Z_\rho \rho \quad \sigma \rightarrow \hat{\sigma} = Z_\sigma \sigma \quad (39)$$

with

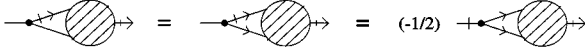
$$Z_\rho = 1 - \frac{u}{8\epsilon} (3+w) + O(u^2), \quad (40a)$$

$$Z_\sigma = 1 - \frac{u}{16\epsilon} [3(w+w^{-1})+2] + O(u^2), \quad (40b)$$

where

$$A_\epsilon g^2 / \rho^{3/2} = u \mu^\epsilon. \quad (40c)$$

In Eq. (40c),  $\mu$  is an external momentum scale and  $A_\epsilon = \Gamma(1-\epsilon/2)/[(1+\epsilon)(4\pi)^{d/2}]$  is a geometrical factor that has been introduced for convenience. The renormalization factors  $\tilde{Z}_1$  and  $Z_1$  are calculated at one-loop order in Appendix A with result

FIG. 1. Perturbative contributions to  $\Gamma_{1,1}$ .

$$\tilde{Z}_1 = 1 + \frac{w-1}{4} \frac{u}{\epsilon} + O(u^2) \quad Z_1 = 1 + O(u^2). \quad (41)$$

We now show that at the stable fixed point  $w=1$  the theory is free of surface singularities, i.e.,  $\tilde{Z}_1=Z_1=1$ . The idea for the proof is presented graphically in Fig. 1. The hatched circle represents the sum of all connected Feynman graphs with a single external  $\tilde{s}$  leg and two  $s$  legs. Together with the three-point vertex representing the bulk coupling  $(\lambda g/2)(\partial_{\parallel}\tilde{s})s^2$  it gives the sum of all perturbative contributions to the (amputated) Green function  $G_{1,1}$ . Lines with or without an arrow denote Dirichlet propagators or correlators while a short line perpendicular to a propagator-correlator line indicates a derivative with respect to  $z$ . By naive power counting one expects that the integrations associated with the first Feynman graph diverge as  $\Lambda_{\parallel}$ . Since the breaking of translational invariance reduces the degree of divergency by unity [1] the graph should give rise to logarithmic surface divergencies. However, Fig. 1 shows that the graph may be replaced by a different diagram with only logarithmic bulk singularities (and thus without surface divergencies).

The first equality in Fig. 1 is justified if we assume that the time variable carried by the leftmost vertex is smaller than the time variables associated with the vertices in the bubble. In this case we can replace the first correlator by a propagator [Eq. (12)]. In fact, if we assume that an *internal* vertex carries the earliest time argument the graph vanishes due to the Dirichlet boundary conditions (see Fig. 2). The second equality in Fig. 1 follows from integration by parts with respect to the  $z$  variable of the first vertex:

$$\int_0^{\infty} dz s \tilde{s} \partial_z \tilde{s} = \int_0^{\infty} dz s \frac{1}{2} \partial_z (\tilde{s}^2) = -\frac{1}{2} \int_0^{\infty} dz \tilde{s}^2 \partial_z s,$$

where we have used the Dirichlet boundary conditions. Since the  $z$  derivative now acts on an external line the degree of divergency is reduced by unity. The remaining logarithmic bulk divergency is canceled by the renormalization of  $\rho$ .

### B. Density profile at one-loop order

In order to compute the lowest order correction to the mean field profile we insert the ansatz

$$\Phi(z) = \Phi_{\text{mf}}(z) + \Phi_1(z) \quad (42)$$

into Eq. (22) and retain only the terms linear in  $\Phi_1(z)$ . The resulting differential equation



FIG. 2. The vertex is assumed to carry the earliest time argument. In this case the integration over  $z$  vanishes due to the Dirichlet boundary conditions.

$$\rho \Phi_1'(z) + \frac{1}{2} g [C(z) - C_{\text{bulk}} + 2\Phi_{\text{mf}}(z)\Phi_1(z)] = 0 \quad (43)$$

has the solution

$$\begin{aligned} \Phi_1(z) &= -\frac{g}{2\rho} \int_0^z dz' \exp\left(-\frac{g}{\rho} \int_{z'}^z \Phi_{\text{mf}}(y) dy\right) [C(z') - C_{\text{bulk}}] \\ &= -\frac{g}{2\rho} \int_0^z dz' \left(\frac{z'+a}{z+a}\right)^2 [C(z') - C_{\text{bulk}}], \end{aligned} \quad (44)$$

where  $a=2\rho/(g\Phi_0)$ . At one-loop order it is sufficient to calculate the density fluctuation  $C(z)$  in a Gaussian approximation. Since this function depends on the density profile we have to generalize the results of the previous subsection to the case  $\Phi(z) \neq 0$ . For this purpose we split the field  $s(\mathbf{r})$  into its mean value  $\Phi(z)$  and a fluctuating part  $\phi(\mathbf{r})$ , which leads to the dynamic functional

$$\mathcal{J}_b[\tilde{\phi}, \Phi + \phi] = \mathcal{J}_0[\tilde{\phi}; \Phi] + \mathcal{J}_G[\tilde{\phi}, \phi; \Phi] + \mathcal{J}_{\text{im}}[\tilde{\phi}, \phi], \quad (45)$$

where

$$\mathcal{J}_0[\tilde{\phi}; \Phi] = \int dt \int_V d^d r \lambda (\partial_{\parallel} \tilde{\phi}) [\rho \Phi'(z) + \frac{1}{2} g \Phi(z)^2], \quad (46)$$

$$\begin{aligned} \mathcal{J}_G[\tilde{\phi}, \phi; \Phi] &= \int dt \int_V d^d r \{ \tilde{\phi} \partial_t \phi + \lambda [(\partial_{\perp} \tilde{\phi})(\partial_{\perp} \phi) \\ &\quad + \rho (\partial_{\parallel} \tilde{\phi})(\partial_{\parallel} \phi) + g \Phi(z) (\partial_{\parallel} \tilde{\phi}) \phi - (\partial_{\perp} \tilde{\phi})^2 \\ &\quad - \rho (\partial_{\parallel} \tilde{\phi})^2] \}, \end{aligned} \quad (47)$$

and

$$\mathcal{J}_{\text{im}}[\tilde{\phi}, \phi] = \int dt \int_V d^d r \frac{1}{2} \lambda g (\partial_{\parallel} \tilde{\phi}) \phi^2. \quad (48)$$

(Throughout this subsection we assume  $w=1$ , i.e.,  $\sigma=\rho$ .) The Gaussian propagator satisfies the differential equations

$$[\kappa^2 - \rho \partial_z^2 - g \partial_z \Phi(z)] \hat{G}_{\mathbf{q}_{\perp}, \omega}(z, z') = \frac{1}{\lambda} \delta(z - z'), \quad (49a)$$

$$[\kappa^2 - \rho \partial_z^2 + g \Phi(z) \partial_z] \hat{G}_{\mathbf{q}_{\perp}, \omega}(z', z) = \frac{1}{\lambda} \delta(z - z'), \quad (49b)$$

which follow from the functional  $\mathcal{J}_G$  (47). The solution for  $\Phi(z) = \Phi_{\text{mf}}(z)$  may be written in the form

$$\begin{aligned} \hat{G}_{\mathbf{q}_{\perp}, \omega}(z, z') &= \frac{1}{2\lambda \sqrt{\rho \kappa}} \frac{1}{\xi^2} \left[ \frac{f_+(\xi_0)}{f_-(\xi_0)} f_-(\xi <) \right. \\ &\quad \left. - f_+(\xi <) \right] f_-(\xi >), \end{aligned} \quad (50)$$

where

$$f_{\pm}(\xi) = (1 \mp \xi) \exp(\pm \xi) \quad (51)$$

and

$$\zeta = \kappa(z+a)/\sqrt{\rho}, \quad \zeta_0 = \kappa a/\sqrt{\rho}, \quad (52a)$$

$$\zeta_{>} = \kappa(\max\{z, z'\} + a)/\sqrt{\rho}, \quad (52b)$$

$$\zeta_{<} = \kappa(\min\{z, z'\} + a)/\sqrt{\rho}. \quad (52c)$$

To obtain the correlator one has to calculate the integral

$$\begin{aligned} \hat{C}_{\mathbf{q}_\perp, \omega}(z, z') &= 2\lambda \int_0^\infty dy \hat{G}_{\mathbf{q}_\perp, \omega}(z, y) \\ &\quad \times (q_\perp^2 + \rho \partial_y^2) \hat{G}_{\mathbf{q}_\perp, -\omega}(z', y). \end{aligned} \quad (53)$$

We use Eqs. (49) to simplify the integral by eliminating one of the two  $y$  derivatives in (53). The result

$$\begin{aligned} \hat{C}_{\mathbf{q}_\perp, \omega}(z, z') &= \hat{G}_{\mathbf{q}_\perp, -\omega}(z', z) + \hat{G}_{\mathbf{q}_\perp, \omega}(z, z') - \lambda g \int_0^\infty dy \Phi_{\text{mf}} \\ &\quad \times (y) \partial_y [\hat{G}_{\mathbf{q}_\perp, \omega}(z, y) \hat{G}_{\mathbf{q}_\perp, -\omega}(z', y)] \end{aligned} \quad (54)$$

can be further simplified by integration by parts since

$$\hat{G}_{\mathbf{q}_\perp, -\omega}(z', y) \partial_y \Phi_{\text{mf}}(y) = \hat{G}_{\mathbf{q}_\perp, -\omega}(y, z') \partial_{z'} \Phi_{\text{mf}}(z'). \quad (55)$$

This manipulation leads to

$$\begin{aligned} \hat{C}_{\mathbf{q}_\perp, \omega}(z, z') &= \hat{G}_{\mathbf{q}_\perp, -\omega}(z', z) + \hat{G}_{\mathbf{q}_\perp, \omega}(z, z') \\ &\quad - \frac{2\lambda\rho}{(z'+a)^2} \int_0^\infty dy \hat{G}_{\mathbf{q}_\perp, \omega}(z, y) \hat{G}_{\mathbf{q}_\perp, -\omega}(y, z'). \end{aligned} \quad (56)$$

The function  $C(z)$  describing the Gaussian density fluctuations is given by the integral of  $\hat{C}_{\mathbf{q}_\perp, \omega}(z, z)$  over  $\mathbf{q}_\perp$  and  $\omega$ . Using the semigroup property

$$G_{\mathbf{q}_\perp}(z, z'; t_1 + t_2) = \int_0^\infty dy G_{\mathbf{q}_\perp}(z, y; t_1) G_{\mathbf{q}_\perp}(y, z'; t_2) \quad (57)$$

of the propagator in  $(\mathbf{q}_\perp, z, t)$  representation we obtain

$$C(z) = C_{\text{bulk}} - \frac{\lambda\rho}{(z+a)^2} \int_{\mathbf{q}_\perp} \hat{G}_{\mathbf{q}_\perp, \omega=0}(z, z), \quad (58)$$

where

$$C_{\text{bulk}} = 2 \int_{\mathbf{q}_\perp, \omega} \text{Re}[\hat{G}_{\mathbf{q}_\perp, \omega}(z, z)] \propto \Lambda_\perp^{d-1} \Lambda_\parallel \quad (59)$$

is independent of  $z > 0$ . The straightforward evaluation of the  $\mathbf{q}_\perp$  integral in (58) yields

$$\begin{aligned} C(z) - C_{\text{bulk}} &= -\frac{\Gamma(1-\epsilon/2)}{(4\pi)^{d/2}} \frac{2\rho^{(1-\epsilon)/2}}{(z+a)^4} z^\epsilon \left[ \frac{(z+a)^2}{\epsilon} - \frac{z^2}{1+\epsilon/2} \right. \\ &\quad \left. - \frac{az}{\Gamma(1-\epsilon)} \int_0^\infty dy \frac{y^{-\epsilon}}{1+ay/(2z)} e^{-y} \right]. \end{aligned} \quad (60)$$

To compute the one-loop correction to the profile one has to insert (60) into Eq. (44), giving

$$\Phi_1(z) = \frac{\Gamma(1-\epsilon/2)}{(4\pi)^{d/2}} \frac{g\mu^{-\epsilon}\rho^{-1/2}}{(z+a)^2} \left( \frac{z(\mu z/\sqrt{\rho})^\epsilon}{\epsilon(1+\epsilon)} - F(z; a, \epsilon) \right) \quad (61)$$

with

$$\begin{aligned} F(z; a, \epsilon) &= (\mu/\sqrt{\rho})^\epsilon \int_0^z dz' \frac{z'^{2+\epsilon}}{(z'+a)^2} \\ &\quad \times \left( \frac{1}{1+\epsilon/2} + \frac{2}{\Gamma(1-\epsilon)} \int_0^\infty dy \frac{y^{-\epsilon}}{y+2z'/a} e^{-y} \right). \end{aligned} \quad (62)$$

In the limit  $\epsilon \rightarrow 0$  we have

$$\begin{aligned} F(z; a, \epsilon=0) &= z \left( 1 + \frac{a}{z+a} \right) - a \left[ \gamma + \ln \left( \frac{2z}{a} \right) \right. \\ &\quad \left. - \frac{z-a}{z+a} \exp \left( \frac{2z}{a} \right) \text{E}_1 \left( \frac{2z}{a} \right) \right], \end{aligned} \quad (63)$$

where  $\gamma = 0.5772, \dots$  is Euler's constant and  $\text{E}_1(x)$  denotes the exponential integral [29]. Equation (60) shows that the perturbation series is not well defined for  $\epsilon = 0$ . This uv divergency is canceled by the renormalization of the diffusion constant  $\rho$  in the mean field profile which leads to

$$\begin{aligned} \Phi(z) &= \frac{2\rho}{g} \frac{1}{z+a} \left[ 1 + \frac{u}{2(z+a)} \left( \frac{z}{\epsilon} [(\mu z/\sqrt{\rho})^\epsilon - 1] \right. \right. \\ &\quad \left. \left. - (1+\epsilon)F(z; a, \epsilon) \right) \right] + O(2\text{-loop}). \end{aligned} \quad (64)$$

Since the coupling constant  $g$  is relevant below two dimensions we may not use the perturbation series at finite order directly to study the asymptotic behavior of the profile. In the following subsection we improve the one-loop result Eq. (64) by a renormalization group analysis.

### C. Scaling

The renormalization group transformation allows us to map the profile for large values of  $z$  (compared to microscopic length scales) to length scales which are accessible to perturbative calculations. Since the bare coupling coefficients are independent of the momentum scale  $\mu$  the profile satisfies (for  $w=1$ ) the renormalization group equation

$$[\mu \partial_\mu + \beta(u) \partial_u + \zeta(u)(\rho \partial_\rho + a \partial_a)] \Phi(z; u, \rho, a; \mu) = 0, \quad (65)$$

where [11]

$$\zeta(u) = \frac{1}{\rho} \mu \frac{d}{d\mu} \Big|_{\rho} = -\frac{1}{2} u + O(u^2) \quad (66)$$

and

$$\beta(u) = \mu \frac{d}{d\mu} \Big|_0 u = u[-\epsilon - \frac{3}{2}\zeta(u)] \quad (67)$$

are defined as derivatives at fixed bare parameters. Equation (65) can be solved by the method of characteristics, with the solution

$$\Phi(z; u, \rho, a; \mu) = \Phi(z; \bar{u}(l), \rho X(l), aX(l); \mu l) \quad (68)$$

and associated characteristics

$$l \frac{d}{dl} \bar{u}(l) = \beta(\bar{u}(l)), \quad l \frac{d}{dl} \ln X(l) = \zeta(\bar{u}(l)). \quad (69)$$

Using dimensional analysis and Eq. (68) it is easy to show that the profile satisfies the relation

$$\begin{aligned} \Phi(z; u, \rho, a; \mu) &= (\mu l)^{d/2} [\rho X(l)]^{-1/4} \\ &\times \Phi(\mu l z [\rho X(l)]^{-1/2}; \bar{u}(l), 1, \mu l a [X(l)/\rho]^{1/2}; 1). \end{aligned} \quad (70)$$

For small  $l$  the scale dependent coupling coefficient  $\bar{u}(l)$  tends to the fixed point  $u_\star = (4/3)\epsilon + O(\epsilon^2)$  and the function  $X(l)$  is asymptotically proportional to a power of  $l$ , namely,

$$X(l) \approx X_0 l^{-2\eta}, \quad \eta = -\frac{1}{2} \zeta(u_\star) = \frac{2-d}{3}, \quad (71)$$

where  $X_0$  is a nonuniversal scale factor. By choosing the value

$$l = [(X_0 \rho)^{1/2} / (\mu z)]^{3/(5-d)} \quad (72)$$

for the flow parameter we obtain for  $z \gg \sqrt{\rho} \mu^{-1}$  the scaling form

$$\Phi(z, \Phi_0) = \Phi_0 \Xi(B \Phi_0 z^{(d+1)/(5-d)}) \quad (73)$$

with the scaling function

$$\Xi(x) = x^{-1} \Phi(1; u_\star, 1, \alpha x^{-1}; 1) \quad (74)$$

[where  $\alpha = (4A_\epsilon/u_\star)^{1/2}$ ] and the nonuniversal constant

$$B = [(\rho X_0)^{1/2} \mu^{\epsilon/3}]^{-3(d-1)/[2(5-d)]}. \quad (75)$$

To derive Eq. (73) we have used the identity

$$A_\epsilon g^2 = u_\star (\rho X_0)^{3/2} \mu^\epsilon, \quad (76)$$

which follows from the flow equations (69) and the initial condition (40c) to express  $a$  in terms of  $\Phi_0$ ,  $\rho$ ,  $u_\star$ ,  $X_0$ , and  $\mu$ . Note that for  $d=1$  there is no adjustable parameter in Eq. (73) since  $B$  is unity in this case. The scaling behavior of the one-dimensional profile was first predicted by Krug [13].

The reciprocal of the scaling function at first order in  $\epsilon$  is given by

$$\Xi(x)^{-1} = 1 + \frac{x}{\alpha} \left( 1 + \frac{2\epsilon}{3} F(1; \alpha/x, \epsilon=0) \right) + O(\epsilon^2). \quad (77)$$

In order to compare this result with the exact solution found in Refs. [14,15] we have to take the continuum limit of the exact profile. In Appendix B we show that this limit leads to

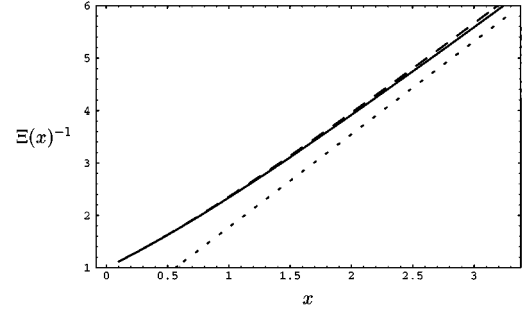


FIG. 3. Solid curve: scaling function  $\Xi_{\text{exact}}(x)^{-1}$  calculated from the exact profile; broken curve: first order in  $\epsilon$ ; dotted line: asymptotic behavior of  $\Xi_{\text{exact}}(x)^{-1}$  for  $x \rightarrow \infty$ .

$$\Xi_{\text{exact}}(x) = \exp(x^2) \text{erfc}(x), \quad (78)$$

where  $\text{erfc}(x) = (2/\sqrt{\pi}) \int_x^\infty dy \exp(-y^2)$  denotes the error function [29]. The exact scaling function and the one-loop approximation (77) for  $\epsilon=1$  are depicted in Fig. 3. The asymptotic form

$$\Xi_{\text{exact}}(x)^{-1} \approx \sqrt{\pi} x \quad \text{for } x \rightarrow \infty \quad (79)$$

of the scaling function can be compared with the asymptotic behavior

$$\Xi(x)^{-1} \approx \frac{1}{\alpha} \left[ 1 + \frac{2}{3} \epsilon + O(\epsilon^2) \right] x \approx 1.086 \sqrt{\pi} x \quad (80)$$

of the  $\epsilon$  expansion at first order for  $\epsilon=1$ . Here and in Fig. 3 we have used the geometric prefactor  $\alpha = (4A_\epsilon/u_\star)^{1/2}$  directly for  $\epsilon=1$  (i.e., without expansion in  $\epsilon$ ) since this procedure is empirically known to give the best numerical results [11].

By a different choice of the flow parameter it is possible to eliminate the  $O(\epsilon)$  contribution to the profile completely. The corresponding value of  $l$  is defined by the equation [see (64) and (70)]

$$\begin{aligned} \mu l z [\rho X(l)]^{-1/2} \ln \{ \mu l z [\rho X(l)]^{-1/2} \} \\ - F(\mu l z [\rho X(l)]^{-1/2}; \mu l a [X(l)/\rho]^{1/2}, \epsilon=0) = 0. \end{aligned} \quad (81)$$

However, this modification has only a small effect on the scaling function  $\Xi(x)^{-1}$  at one-loop order.

We conclude this section with the discussion of logarithmic corrections to the mean field profile in two dimensions. For  $\epsilon=0$ ,  $w=1$  the scale dependent coupling coefficient  $\bar{u}(l)$  is the solution of the differential equation

$$l \frac{d}{dl} \bar{u}(l) = \beta[\bar{u}(l)] = \frac{3}{4} \bar{u}(l)^2 + O[\bar{u}(l)^3] \quad (82)$$

with the initial condition  $\bar{u}(1) = u$ . For  $l \rightarrow 0$  we find the asymptotic expression

$$\bar{u}(l) = \frac{4}{3} [\ln(1/l)]^{-1} \left[ 1 + O\left( \frac{\ln \ln(1/l)}{\ln(1/l)} \right) \right]. \quad (83)$$



A straightforward calculation yields the asymptotic behavior of the characteristic  $X(l)$ ,

$$\ln X(l) = \int_1^{l'} \frac{dl''}{l''} \zeta(\bar{u}(l'')) \approx \frac{2}{3} \ln \ln(1/l) + \text{const} \quad (84)$$

for  $l \rightarrow 0$ , i.e.,  $X(l) \sim [\ln(1/l)]^{2/3}$ . In order to sum the logarithmic singularities in the perturbation series we set  $l = \sqrt{\rho}/(\mu z)$  in Eq. (68) and obtain

$$\Phi(z, \Phi_0) = \Phi_0 [1 + \text{const} \times \Phi_0 z (\ln(\mu z / \sqrt{\rho}))^{-2/3}]^{-1} \quad (85)$$

$$\sim \frac{(\ln z)^{2/3}}{z} \quad \text{for } z \rightarrow \infty. \quad (86)$$

This result is valid on length scales that are large compared to microscopic distances, i.e.,  $z \gg \Lambda_{\parallel}^{-1}$ .

## V. SUMMARY AND OUTLOOK

Until now most analytical studies of driven diffusion deal with closed systems. To maintain a steady state with a non-vanishing current one usually imposes periodic boundary conditions in the direction parallel to the driving force [10]. Since in real physical systems the particles are fed into the system by a reservoir the effects of boundaries are of great interest. Using a field theoretic renormalization group approach we have calculated the density profile in a semi-infinite DDS with a particle reservoir at the boundary. For  $d=1$  our result is in fair agreement with the exact solution for the totally asymmetric exclusion model [14,15]. Equation (85) for the two-dimensional profile is a result that may be checked by computer simulations.

We plan to extend the present work along several lines:

(1) We have studied the effect of a particle source (or sink) at  $z=0$  assuming a second reservoir to be located at  $z=L$  with  $L \rightarrow \infty$ . One would also like to consider the interplay of the reservoirs for finite  $L$  [14,15].

(2) In a system with quenched disorder and periodic boundary conditions a particle is subjected to the same random potential after every passage through the system. In this way the periodicity produces long-range correlations that make it difficult to compare simulation results with theoretical predictions that assume uncorrelated disorder [17]. To circumvent this problem it would be desirable to consider disordered DDS with open boundary conditions.

(3) In the case of noncritical DDS the scaling behavior (73) of the profile can be described by a single exponent, namely, the bulk exponent  $\eta = (2-d)/3$ . This is due to the fact that the surface density  $\Phi_0$  has the same scaling dimension as the bulk density. We expect that this is no longer true if the system is at its critical point. In general the surface critical behavior is governed by new exponents that cannot be expressed in terms of bulk exponents.

(4) Another line of extension would be to consider boundaries parallel to the driving force.

(5) In Sec. IV we have shown that for  $d > 1$  the flat profile  $\Phi(z) = 0$  is only stationary in the case of detailed balance and in the limit of high temperature and strong field. In the more general case of a lattice gas at finite temperature the

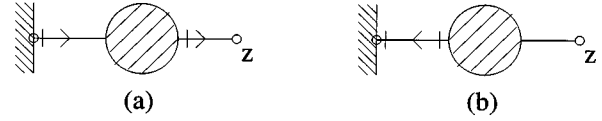


FIG. 4. Perturbative contributions to the (a) surface-bulk and the (b) bulk-surface response function. Here the hatched circle represents the integral kernel (A5).

long-range correlations in the bulk give rise to an inhomogeneous steady state even for  $\Phi_0 = \Phi_{\text{bulk}}$ . [It was already pointed out by Zia and Schmittmann [19] that the coupling coefficient  $(1-w)$  that is responsible for the long-range correlations is dangerously irrelevant for  $1 < d \leq 2$ .] This effect should lead to a modification of the maximum current principle [Eq. (1)]. Finally, it would be interesting to study the formation of the stationary density profile starting from a homogeneous initial state.

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## APPENDIX A: SURFACE SINGULARITIES AT ONE-LOOP ORDER

In Sec. IV we have shown that for  $w=1$  the perturbation expansion is free of surface singularities. In this Appendix we consider the general case  $w \geq 0$ .

The Fourier transform of the surface-bulk response function

$$\chi_{sb}(\mathbf{r}_{\perp}, z; t) = \langle s(\mathbf{r}_{\perp}, z; t) \rho \partial_n \bar{s}(\mathbf{0}, 0) \rangle \quad (A1)$$

and the bulk-surface response function

$$\chi_{bs}(\mathbf{r}_{\perp}, z; t) = \langle \rho \partial_n s(\mathbf{r}_{\perp}, t) \bar{s}(\mathbf{0}, z; 0) \rangle \quad (A2)$$

may be written in the form

$$\begin{aligned} \hat{\chi}_{sb}(\mathbf{q}_{\perp}, \omega, z) &= \rho \partial_{z'} \hat{G}_{\mathbf{q}_{\perp}, \omega}(z, z')|_{z'=0} \\ &+ \int_0^{\infty} dy \int_0^{\infty} dy' K_{\mathbf{q}_{\perp}, \omega}(y, y') \partial_y \hat{G}_{\mathbf{q}_{\perp}, \omega}(z, y) \\ &\times \rho \partial_{z'} \hat{G}_{\mathbf{q}_{\perp}, \omega}(y', z')|_{z'=0} \end{aligned} \quad (A3)$$

and

$$\begin{aligned} \hat{\chi}_{bs}(\mathbf{q}_{\perp}, \omega, z) &= \rho \partial_{z'} \hat{G}_{\mathbf{q}_{\perp}, \omega}(z', z)|_{z'=0} \\ &+ \int_0^{\infty} dy \int_0^{\infty} dy' K_{\mathbf{q}_{\perp}, \omega}(y', y) \partial_y \hat{G}_{\mathbf{q}_{\perp}, \omega}(y, z) \\ &\times \rho \partial_{z'} \hat{G}_{\mathbf{q}_{\perp}, \omega}(z', y')|_{z'=0}, \end{aligned} \quad (A4)$$

respectively, where  $\hat{G}_{\mathbf{q}_{\perp}, \omega}(z, z')$  is the free propagator (32) in Fourier representation (see Fig. 4). At one-loop order the

kernel  $K_{\mathbf{q}_\perp, \omega}(z, z')$  is given by

$$K_{\mathbf{q}_\perp, \omega}(z, z') = (\lambda g)^2 \int_{\mathbf{k}_\perp, \nu} [\partial_{z'} \hat{G}_{\mathbf{q}_\perp - \mathbf{k}_\perp, \omega - \nu}(z, z')] \times \hat{C}_{\mathbf{k}_\perp, \nu}(z, z') + O(g^4). \quad (\text{A5})$$

Note that  $K_{\mathbf{q}_\perp, \omega}(z, z')$  has to be treated as a distribution since the calculation of response functions involves integrations of  $K_{\mathbf{q}_\perp, \omega}(z, z')$  with respect to  $z$  and  $z'$ .

Before we calculate the singular part of the kernel explicitly it is helpful to discuss the general form of the divergencies one has to expect. First,  $K_{\mathbf{q}_\perp, \omega}(z, z')$  has a nonintegrable (for  $d=2$ ) singularity if  $z \rightarrow z'$ . This divergency occurs in

the translationally invariant bulk theory as well as in the semi-infinite model and has to be cured by a renormalization of  $\rho$ . The form of this singularity follows from dimensional analysis and the isotropy in the directions parallel to the surface. Since  $K_{\mathbf{q}_\perp, \omega}(z, z') \sim \lambda \mu^2$  and  $z \sim \sqrt{\rho} \mu^{-1}$  the (dimensionally regularized) bulk singularity contained in  $K_{\mathbf{q}_\perp, \omega}(z, z')$  is proportional to  $\lambda \rho \delta'(z - z')$ . The breaking of translational invariance by the boundary generates additional divergencies for  $z, z' \rightarrow 0$  [1]. Upon dimensional regularization they are proportional to  $\lambda \rho \delta(z) \delta(z')$ .

In order to compute the coefficients of the distributions we apply the kernel to exponential test functions. At lowest order in  $g^2$  we have

$$\int_0^\infty dz \int_0^\infty dz' K_{\mathbf{q}_\perp, \omega}(z, z') e^{-\beta z - \beta' z'} = \frac{\lambda g^2 \mu^{-\epsilon}}{\epsilon \sqrt{\rho}} \frac{\Gamma(1 + \epsilon/2)}{(4\pi)^{d/2}} \left[ \frac{3+w}{8} \frac{\beta'}{\beta + \beta'} + \frac{w-1}{8} \right] + O(\epsilon^0, g^4). \quad (\text{A6})$$

The inverse Laplace transform of this expression with respect to  $\beta$  and  $\beta'$  is given by

$$K_{\mathbf{q}_\perp, \omega}(z, z') = \frac{\lambda g^2 \mu^{-\epsilon}}{\epsilon \sqrt{\rho}} \frac{\Gamma(1 + \epsilon/2)}{(4\pi)^{d/2}} \left[ \frac{3+w}{8} \delta'(z'|z) + \frac{w-1}{8} \delta(z) \delta(z') \right] + O(\epsilon^0, g^4), \quad (\text{A7})$$

where we have introduced the definition

$$\int_0^\infty dz' \delta'(z'|z) f(z') = -f'(z). \quad (\text{A8})$$

[This distribution is equivalent to  $\delta'(z' - z)$  if we define  $\delta(-z) \equiv 0$  for  $z \geq 0$ .] It is now straightforward to calculate the singular parts of the response functions at one-loop order. Inserting (A7) into Eqs. (A3) and (A4) we find

$$\hat{\chi}_{sb}(\mathbf{q}_\perp, \omega, z) = \frac{1}{\lambda} \left[ 1 + \frac{u}{\epsilon} \left( \frac{w-1}{8} + \frac{3+w}{16} \frac{\kappa z}{\sqrt{\rho}} \right) + O(\epsilon^0, u^2) \right] e^{-\kappa z / \sqrt{\rho}} \quad (\text{A9})$$

and

$$\hat{\chi}_{bs}(\mathbf{q}_\perp, \omega, z) = \frac{1}{\lambda} \left[ 1 + \frac{u}{\epsilon} \left( \frac{3+w}{16} \frac{\kappa z}{\sqrt{\rho}} + O(\epsilon^0, u^2) \right) \right] e^{-\kappa z / \sqrt{\rho}}. \quad (\text{A10})$$

To obtain renormalized response functions one has to express the coupling coefficients  $\rho$  and  $\sigma$  by their renormalized counterparts [see Eq. (39)] and absorb the remaining  $\epsilon$  poles into renormalizations of surface fields, i.e.,

$$[\hat{\chi}_{sb}]_R = \bar{Z}_1^{-1/2} \hat{\chi}_{sb}, \quad [\hat{\chi}_{bs}]_R = Z_1^{-1/2} \hat{\chi}_{bs} \quad (\text{A11})$$

with

$$\bar{Z}_1 = 1 + \frac{w-1}{4} \frac{u}{\epsilon} + O(u^2), \quad Z_1 = 1 + O(u^2). \quad (\text{A12})$$

The renormalization factor  $Z_1$  is unity at every order of the perturbation series since the primitive surface divergencies in  $K_{\mathbf{q}_\perp, \omega}(z, z')$  are proportional to  $\delta(z) \delta(z')$ . Due to the Dirichlet boundary conditions this distribution has no effect in Eq. (A4).

## APPENDIX B: THE EXACT PROFILE IN THE CONTINUUM LIMIT

One of the simplest examples of a DDS is the totally asymmetric exclusion model in one dimension [30]. For this model with nearest neighbor hopping and open boundary conditions the density profile has been calculated exactly by Schütz and Domany [14] and Derrida *et al.* [15].

The model is defined on a chain of  $N$  sites each of which is either occupied by a single particle or empty. At each time step one chooses a random lattice site  $p$ ,  $1 \leq p < N$ . If this site is occupied by a particle ( $n_p = 1$ ) and its right neighbor site is empty ( $n_{p+1} = 0$ ), the particle will jump from  $p$  to  $p+1$ . In the model with open boundary conditions particles are injected with rate  $\alpha$  at the left boundary and leave the system with rate  $\beta$  at the right boundary.

For  $\alpha, \beta \geq 1/2$  the bulk density takes the critical value  $n_c = 1/2$  and the system carries the maximum current. Since we wish to compare the  $\epsilon$  expansion of Sec. IV with the exact profile we let  $N \rightarrow \infty$  in order to eliminate the influence of the right boundary. Using the formulas given in the Appendix of Ref. [14] we arrive at the particle density

$$\langle n_p \rangle = \frac{1}{2} + \frac{2}{4^p} \frac{(2p-2)!}{(p-1)!^2} \left[ 1 - \frac{1}{2p(2\alpha-1)^2} F\left( \frac{3}{2}; p+1; -\frac{4\alpha(1-\alpha)}{(2\alpha-1)^2} \right) \right] \quad (\text{B1})$$

for  $\alpha, \beta \geq 1/2$  and  $N \rightarrow \infty$ , where  $F(a, b; c; z)$  is a hypergeometric function [29].

To compute the scaling function  $\Xi_{\text{exact}}(x)$  from the microscopic profile one has to take an appropriate continuum limit. In addition to the distance from the boundary the injection rate  $\alpha$  provides the only macroscopic length scale  $\xi_\alpha = (2\alpha-1)^{-2}$  in the model. We therefore measure  $p$  in units of  $\xi_\alpha$  and let  $p, \xi_\alpha \rightarrow \infty$  at fixed  $x^2 = p/\xi_\alpha$ . For this purpose it is convenient to use the integral representation

$$\begin{aligned} F\left[1, \frac{3}{2}; p+1; -(\xi_\alpha-1)\right] &= p \int_0^1 (1-t)^{p-1} [1 + (\xi_\alpha-1)t]^{-3/2} dt \quad (\text{B2}) \\ &= \int_0^p \exp[(p-1)\ln(1-\tau/p)] [1 + (\xi_\alpha-1)\tau/p]^{-3/2} d\tau \quad (\text{B3}) \end{aligned}$$

to show that

$$\lim_{p, \xi_\alpha \rightarrow \infty} F\left[1, \frac{3}{2}; p+1; -(\xi_\alpha-1)\right] = \int_0^\infty e^{-\tau} (1 + \tau/x^2)^{-3/2} d\tau. \quad (\text{B4})$$

For large  $p$  the prefactor in Eq. (B1) is given by

$$\frac{2}{4^p} \frac{(2p-2)!}{(p-1)!^2} = \frac{1}{2\sqrt{\pi p}} [1 + O(1/p)]. \quad (\text{B5})$$

A straightforward calculation now yields in the continuum limit defined above

$$\langle \sigma_p \rangle = \sigma_0 \exp(\sigma_0^2 p) \operatorname{erfc}(\sigma_0 \sqrt{p}), \quad (\text{B6})$$

where we have expressed the occupation numbers in terms of the spin variables  $\sigma_p = 2n_p - 1$ ,  $\sigma_0 = 2\alpha - 1$ .

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